Nonlinear Dynamics, Complexity and Randomness: Algorithmic Foundations*

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14 June 2010

*Many fruitful conversations on the interaction between dynamical systems theory, computability theory and the notion of complexity with Joe McCauley and Stefano Zambelli were decisive in disciplining our own thoughts. Many years of frienshship and friendly conversations with Richard Day and Barkley Rosser have left indelibly warm marks, in the thought processes that have gone into many issues dealt with in this paper, in the increasingly frail mind of the 'senior' author. Naturally, none of these worthies are even remotely responsible for the remaining infelicities.

Forthcoming in: The Journal of Economic Surveys

Abstract

In this paper we make an attempt to describe, discuss and extend a few aspects of the rich mathematical tapestry that can be woven with rigorous notions of nonlinear dynamics, complexity and randomness, in terms of *algorithmic mathematics*. It is a tapestry that we try to weave with economic analysis, economic theory and economic modelling in mind. All three notions – i.e., nonlinear dynamics, complexity and randomness – have a rich conceptual, modelling or analytic tradition in core areas of economic theory, both at the micro and macro levels. It is the algorithmic foundation we try to provide for them that could be considered the novel contribution in this paper. Once the algorithmic foundations are in place, it is, for example, almost natural to consider the famed difficulties of obtaining closed form solutions for nonlinear, complex or random dynamic models in economics almost a trivial vestige of a pre-simulation era in mathematical modelling.

JEL Classification Codes: B 40, C 02, C 60

Key Words: Nonlinear Dynamics, Emergence, Complexity, Macroeconomics, Algorithmic Mathematics.

1. Nonlinear Dynamics, Complexity and Randomness – General Algorithmic Considerations

> "It should be mentioned that [the development of KAM (**K**olmogorov-**A**rnold-**M**oser) theory and the 'hyperbolic revolution'] vary with particular vigour in the theory of dynamical systems. In their 'purest' form they occur in differential dynamics as quasi-periodicity, for which a certain regularity is characteristic, and as hyperbolicity, which is connected with those phenomena that are descriptively named *'quasi-randomness,' 'stochasticity,' or 'chaos'*. ... Kolmogorov .... was the only one ... who made an equally large contribution to the study of both regular and chaotic motions (but in different parts of the theory of dynamical systems - the differential and the ergodic). ...[T]he *differences between regularity and chaos* do not simply lie on the surface, but *are buried deep within*, so that it is difficult for even the best specialists to switch from one to the other.
>
> The question arises: Is there some other `sufficiently substantial` class of motions that could occupy an intermediate position between the quasi-periodic and the hyperbolic motions (or, perhaps, lie somewhere to the side of both)? Could horocyclic flows and (or) nilflows (or, *perhaps something of the sort that we do not yet know about*) play this role?
>
> D.V. Anosov, 2006, [2], pp. 2-3; italics added.

Arguably, no one man contributed more to the theoretical foundations and frontiers of the tryptich of nonlinear dynamics, complexity - at least an important variant of it - and randomness, the latter two in algorithmic modes, than Kolmogorov. But even he missed out on two related foundational developments that contributed to the algorithmic unification of dynamics and complexity: the varieties of surprising outcomes that were inspired by the famous Fermi-Pasta-Ulam (**FPU**) computational experiment[1], on the one hand, and the developments in ordinary differential equations (**ODE**) from the point of view of constructive and computable analysis, on the other. In the latter case, the constructive and computable approach to ODEs also had, at least as a by-product, considerations of the computational complexity of the solution procedures that were routinely invoked in their analysis.

There is also the recent development in research at the frontiers of the interface between dynamical systems theory, the theory of numerical analysis and computability (cf., for example, [39]). There are those – like the authors of [4] – who believe that the tradition of computation represented by the theory of numerical analysis is not captured by the notion of effective calculability in computability theory. On the other hand, the classical theory of numerical analysis is squarely

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[1] It is rarely acknowledged that economists were prescient in this line of research via the remarkable analysis of '*coupled markets*' – integrating considerations of interdependence and (nonlinear) dynamics – that was pioneered by Richard Goodwin almost fully a decade before even Fermi, Pasta and Ulam (cf. Goodwin, 1947). In the concluding section of his pioneering analysis, Goodwin observed:

> "To go from two identical markets to *n* nonidentical ones will require the prolonged services of *yet unborn calculating machines*." (ibid, p. 204; italics added).

This classic paper by Goodwin was made famous by Herbert Simon in his own sustained research on causality and *near-decomposability*, eventually also to form the 'backbone' of his analysis of evolution (cf., for example: Simon (1953) and (1997; especially **I.3** & **I.4**)) algorithmic. There is now a systematic approach to the study of dynamical systems by exploiting a 'duality' between the intrinsic dynamics of algorithms and their underpinning in computability theory. If this `duality' is then linked to the algorithmic basis of the classical theory of numerical analysis, much of the *ad hockery* of arbitrary discretization and computation of economic dynamic models can be avoided.

When Roger Penrose wondered *whether the Mandelbrot set is recursive* ([31], p.124, ff), the issue became one of the *algorithmic decidability* of the attractors of a certain class of complex dynamical systems, generated by *iterated function systems* (**IFS**). Although the theory of complex dynamical systems, in the sense of – say – rational maps on the Riemann sphere (see, for example, [25]) are almost as old as 'modern' dynamical systems theory as pioneeringly developed by Poincarè in the late 19th century, interpreting its dynamics algorithmically and linking that vision with computation is of much more recent origin (cf., [14]). These issues of the decidability and computability of IFS continue a noble tradition in computability theory, the birth of which owes much to the valiant attempts made by a series of eminent logicians and mathematicians to encapsulate, formally, certain intuitive notions, such as effectivity, calculability, and so on. They are also related to the classical mathematician's eternal quest for the correct formal definition of, for example, the intuitive notion of continuity. We know now that Bourbaki did not capture the entire intuitive content of continuity by 'his' definition of continuity (cf., [14]). Penrose feels that the *Mandelbrot set* – therefore, also, related IFS that generate *Julia, Fatou* and other similar sets - is *naturally recursive* in an intuitive sense that is not captured by the formal notion of a recursive set as defined in computability theory.

Surely Anosov's thoughts on dynamical systems, a small part of which is quoted above, belongs to this tradition of wondering whether the richness of actually occurring dynamics have all been adequately encapsulated in the existing formalisms of dynamical systems theory.

Whether the economist's easy reliance on harnessing existing formalisms to force economic dynamics into a possibly illegitimate straitjacket of dynamical systems theory or a stochastic process, thereby distorts natural economic processes and dresses them in the clothes of theories that have emasculated the intrinsic dynamics belongs to this tradition or not is one of the main motivations for this paper. But we are aware that this is a question that requires the full force of methodological, epistemological and philosophical enquiry, coupled to experimental and mathematical investigations that need much more space than can be devoted to them in one article. We try to take a few small steps in what we hope is a direction that will ultimately help us face the main question more systematically.

A possibly pernicious line of thought in defining economic complexity by relying on formal (nonlinear) dynamical systems theory is the concentration on defining the notion of dynamic complexity in terms of the attractors of such systems. This way of defining and studying the complexity of economic processes has led to unnecessary reliance on so-called chaotic dynamics and forcing economic assumptions so that a resulting economic dynamics would fulfil the necessary – and, often, also sufficient – conditions to generate such preconceived dynamics. Indeed, this has been the hallmark of nonlinear dynamics in economics, from the earliest stages of nonlinear macrodynamics to the more recent enthusiasms for varieties of complex economic dynamics underpinned by assumptions of fundamental nonlinearities in microeconomics. In all of these attempts at generating seemingly complex economic dynamics, the attention has been squarely and systematically placed on the characterisation of the attractors.

A counter-example to this tradition is to view economic processes as always being in transition regimes[2] and to model the dynamics with definably ultra-long transients towards trivial attractors. To disabuse the economists' practice and belief in concentrating the analysis of economic processes in terms of characterising their attractors, we can cite the noble example of the Goodstein algorithm and the associated dynamical system whose attractor is the trivial set of a zero limit point (cf. [15], [30] and [43]). If this approach to the study of economic processes is made systematic, then nonlinearity, complexity and randomness become the basis of the study of economies eternally in transition. It may eventually become possible to do serious studies of economic processes devoid of the unnecessary excess baggage of steady state equilibria - whether of the deterministic or stochastic variety. Moreover, it will return the theory of economic quantities to its natural domain: the rational numbers.

In spite of these general observations and desiderata, the main contribution of the rest of this paper is confined to a study of a particular question which enables us to bring in nonlinearity, complexity and randomness in algorithmic modes in a consistent and simple way. By starting from an intuitive criterion for complexity, suggested originally by Richard Day and imaginatively, if non-rigorously, advocated and applied by Barkley Rosser, we develop an algorithmic framework within which we are able to underpin the link between complexity, nonlinear dynamics and a notion of randomness. In the next section, therefore, we summarise the intuitive content of the Day-Rosser suggestions for studying and defining dynamic economic complexity. In section three we attempt to model these suggestions in a way that unifies the nonlinear complex dynamics of economic processes algorithmically – and provides a bridge towards the study of its algorithmically random content, too. The final section is – for the moment, at least - purely speculative, going beyond nonlinearity, complexity and randomness, but remaining within the fold of the philosophy, epistemology and methodology of algorithmic mathematics to study the notion of *emergence* in such systems.

[2] We may as well ‘confess’ our adherence to the importance of viewing economic dynamics as `transition dynamics' in twin paradoxical modes: the Austrian ‘traverse dynamics’ and the Keynesian ‘transition dynamics’. The Austrian notion of ‘traverse dynamics’ is a well mined analytical tradition. It was in a famous footnote in Chapter 17 of the GT that Keynes stressed the importance of transition regimes (and made the reference to Hume as the progenitor of the equilibrium concept in economics ([20], p. 343, footnote 3; italics added):

> “[H]ume began the practice amongst economists of stressing the importance of the equilibrium position as compared with the ever-shifting transition towards it, though he was still enough of a mercantilist not to overlook the fact that *it is in the transition that we actually have our being*:…”

2. Nonlinear Dynamics and Dynamic Complexity

> "What one is typically confronted with is some particular physical system whose constituents are governed by perfectly well-understood basic rules. These rules are usually *algorithmic*, in that they can be described in terms of functions simulatable on a computer, and their simplest consequences are mathematically predictable. But although the global behaviour of the system is *determined* by this algorithmic content, it may not itself be recognisably algorithmic. We certainly encounter this in the mathematics, which may be *nonlinear* and not yield the exact solutions needed to retain predictive control of the system. We may be able to come up with a perfectly precise *description* of the system's development which does not have the predictive – or algorithmic – ramifications the atomic rules would lead us to expect."
>
> S. Barry Cooper, 2006, p. 194 ([8]; italics in the original)

Barkley Rosser has opted for what he calls Richard Day's 'broad tent definition' of complexity in his work on *nonlinear, complex* and *emergent* economics – both in microeconomics and macroeconomics – with an underlying focus also on the importance of discontinuities (Rosser, 1999 [36], pp. 170-1; italics added):

> "A 'broad tent' definition, following Richard H. Day (1994) [[10]], is that a dynamical system is complex if it endogenously does *not* tend asymptotically to a fixed point, a limit cycle, or an explosion.
>
> ....
>
> Despite its 'broad tent' nature, this definition does not fit all of what some economists have called complexity. ....
>
> However, Day's (1994) broad-tent definition remains attractive, because it is sufficiently broad that it includes not only most of what is now generally labeled complexity, but also its nonlinear dynamics predecessors... .
>
> Although complexity is a multidisciplinary concept derived from mathematics and physics, the extra complications arising in economics because of the problem of interacting human calculations in decision-making add a layer of complexity that may not exist in other disciplines."

Richard Day's *magnum opus*, **Complex Economic Dynamics** (Vols I & II; [10], [11], in particular, section 25.2) has a consistent approach to dynamic complexity for modelling in economics, again both at the microeconomic and macroeconomic level[3].

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[3] In a recent reflection on *Complex Economic Dynamics*, [12], contrasting his vision with that of the dominant school of macroeconomics, Day observed (p. 204; italics added):

> "A contrasting category of work [to the DSGE methodology] to which my own studies belong views the observed fluctuations and instabilities as being intrinsic to the development process, that economies do not converge to stable stationary situations or to ones of steady uninterrupted economic growth, and that they can be explained by the formal representation of

This approach is summarised concisely as one where the 'definition of complexity' entails (italics added)[4]:

> Types of change that are *not* periodic or balanced and that do *not* converge to a periodic or balanced pattern. Such *paths* are called *complex*. In particular they include processes that involve:

- nonperiodic fluctuations
- overlapping waves
- switches in regime or *structural change*

> These types of changes .... are ubiquitous phenomena in the economics of experience."

In one of his more recent contributions to the subject (Rosser, 2010 [[37]], p.185; italics added) whilst reiterating and finessing the above definition, and reaffirming his commitment to it, he goes beyond and further on the concept of complexity, by considering issues of 'emergent complexity', as well as being much more specific about the role of nonlinearity[5]:

> In general, one advantage of the dynamic definition is that it provides a clearer distinction between systems that are complex and those that are not, although there are some fuzzy zones as well with it.

> *adaptive economizing and market mechanisms that function out-of-equilibrium*."

Day's characterization of dynamic complexity must, therefore, be understood against the backdrop of *adaptively economizing agents functioning in out-of-equilibrium situations*.

4 This characterisation was stated in an e-mail response by Dick Day to a query from the first author for a succinct statement summarising his general vision of dynamic complexity, on 11 February, 2009. In trying to understand Richard Day's visions of dynamic complexity, it must be remembered that he is, together with Herbert Simon, James March, Richard Cyert, Sidney Winter and Richard Nelson, one of the original founders of what was once called `behavioural economics'. What is now called `behavioural economics' is a bastardization of the original, computationally constrained, approach to behaviour by individuals and organisations. It is not without significance that Day is the founding father of the *Journal of Economic Behavior and Organization* - and Rosser is his anointed successor.

5 One of the many starting points for the themes he develops in Rosser (2010) is the concluding conjecture in Velupillai (2008; p. 23; [42]):

> "The `fallacy of composition' that drives a felicitous wedge between micro and macro, between the individual and the aggregate, and gives rise to emergent phenomena in economics, in non-algorithmic ways – as conjectured, originally more than a century and a half ago – by John Stuart Mill ([26]) and George Herbert Lewes ([22]), and codified by Lloyd Morgan in his Gifford Lectures ([23]) – may yet be tamed by unconventional models of computation."

This paper is a step in the direction of showing the feasibility of a formal answer to that conjecture.

Thus, while the definition given above categorizes systems with deterministically endogenous limit cycles as not complex, some observers would say that any system with a periodic cycle is complex as this shows endogenous business cycles in macroeconomics. Others argue that complexity only emerges with aperiodic cycles, the appearance of chaos, or discontinuities associated with bifurcations or multiple basins of attraction or catastrophic leaps or some form of non-chaotic aperiodicity. So, there is a gradation from very simple systems that merely converge to a point or a growth path, all the way to fully aperiodic or discontinuous ones. *In almost all cases, some form of nonlinearity is present in dynamically complex systems* and is the source of the complexity, whatever form it takes. ....

The idea of either emergence or evolution, much less emergent evolution, is not a necessary part of the dynamic conceptualization of complexity, and certainly is not so for the various computational ones.

However, in our opinion 'the idea of emergence', as it appeared in the works of the founding fathers – John Stuart Mill and George Henry Lewes – *can* be given an *algorithmic* interpretation and, hence, either a constructive mathematical or recursion theoretic conceptualization. Thus, any formal characterization of 'the idea of emergence' would have had to have been 'a necessary part of the dynamic conceptualization of complexity', since an algorithm is by definition a dynamic object, particularly in its classic formalization by Alan Turing[6]. So far as we are concerned, every one of the more seriously formal definitions of complexity can, ultimately, be `reduced' to a basis in computability theory or in Brouwerian constructive mathematics. Many of the frontiers of theoretical nonlinear dynamics have become issues on the interface between dynamical systems theory, model theory, constructive analysis, computable analysis and numerical analysis. The dynamical system interpretation of numerical procedures, in turn, links it with issues of decidability and uncomputability, on the one hand, and computational complexity, on the other, in natural settings (cf. Stuart and Humphries, 1996, [[39]] and Blum, et.al [[4]]).

In this next section, we concentrate exclusively on formalizing the notion of dynamic complexity as suggested by Day and made the fulcrum of his many interpretations of complex economic dynamics by Barkley Rosser.

An important caveat, from the point of view of computability theory (or, perhaps, computable economics), must be stated here. The 'broad tent' definition is a kind of *negative criterion* for 'dynamic complexity'. In this 'negative criterion' sense, the search is for a dynamical systems that are recursively enumerable but not recursive, at

[6] Many of the pioneers of computability theory posed the fundamental question to be asked as: `What is a computable function?' and, then, go on to characterise them as a subset of the totality of mathematically definable functions. Turing, on the other hand, emphasised the dynamic underpinnings of any computational activity, Turing (1936-7), p. 245 ([[40]], italics added):

> The real question at issue is "What are the possible processes which can be carried out in computing a number?".

the least. Such systems bristle with algorithmic undecidabilities. To this if we append the slightly finer requirement of the distinction between recursive separability and recursive inseparability, we will also add some of the paradoxes of algorithmic dynamics, such as: computable dynamical systems giving rise to uncomputable outcomes.

3. A Complex (Nonlinear) Dynamical System

> [A] fundamental problem in methodology [is that] the traditional theory of "dynamical systems" is not equipped for dealing with constructive[7] processes. … We seek to solve this impasse by connecting dynamical systems with fundamental research in computer science … … Many failures in domains of biological (e.g., development), cognitive (e.g., organization of experience), social (e.g., institutions), and economic science (e.g., markets) are nearly universally attributed to *some combination of high dimensionality and nonlinearity. Either alone won't necessarily kill you, but just a little of both is more than enough. This, then, is vaguely referred to as complexity.*
>
> Fontana & Buss, 1996 [[13]], pp. 56-7, italics added.

Fortunately the Day-Rosser approach to the characterization of a complex dynamical system eschews any reliance on 'high dimensionality'! In fact, it is easy to show – even constructively – that nonlinearity alone is sufficient to 'kill you' and that 'high dimensionality' is, at best, only an unnecessary adornment. Indeed, an appropriately constructed nonlinear dynamical system, in fact, a piecewise-linear dynamical system, 'connected with fundamental research in computer science' can generate the kind of dynamical complexity characterized by the Day-Rosser definition *even in one-dimension*. The perceptive observation, 'connecting dynamical system with fundamental research in computer science', is crucial in this regard.

The essential features of the Day-Rosser characterization are simply the two workhorses of classic nonlinear endogenous business cycle theories:

- Nonlinearity
- Endogeneity

Even the 'new' twist suggested by Fontana and Buss of 'connecting dynamical systems with fundamental research in computer science', in the context of juxtaposing nonlinear dynamics with complexity, has a respectable pedigree in economic dynamics. Now almost sixty years ago, Richard Goodwin (1951) pointed out [[16], pp.1-2][8]:

> [I]f our methods for determining the successive approximations [to a solution] are made analogous to the structure of economic decisions,

[7]Fontana and Buss are not referring to `constructive' in its mathematical senses of Brouwerian constructive mathematics – or any other variant, such as Bishop-style or Russian constructivism.

[8]However, it must be remembered that Goodwin was thinking of an *analogue computer* when he made this statement.

> then we may regard the sequence of steps as entirely parallel to an actual process of economic dynamics in time. … [And] we may regard economic dynamics as such a series of iterated trial solutions which actually succeed one another at realistically great, regular, intervals of time. (p.1)

and, hence:

> It is entirely permissible to regard the motion of an economy as a process of computing answers to the problems posed to it…. . (p.2)

What we try to accomplish in this section are the following four tasks. First define, for an abstract (one-dimensional) dynamical system, the concept of *universality* via an equivalence with a dynamical system with a suitably initialized *Universal Turing Machine*. Next, show that the attractors of such dynamical systems satisfy the Day-Rosser criterion for dynamic complexity. Then, construct a *minimal dynamical system* capable of computation universality. Finally, going slightly beyond the Day-Rosser aims, we also show how the *algorithmic complexity* of such a minimal dynamical system, capable of computation universality, can be given formal content in terms of the *uncomputable Kolmogorov complexity* of the equivalent *Universal Turing Machine*.

We shall have to assume familiarity with the formal definition of a dynamical system (but, cf. for example, the obvious and accessible classic, [19] or the more modern, [5], for the basic terms and concepts that are assumed here), the necessary associated concepts from dynamical systems theory and all the necessary notions from classical computability theory (for which the reader can, with profit and enjoyment, go to a classic like [35] or, at the frontiers, to [7]). Just for ease of reference the bare bones of relevant definitions for dynamical systems are given below in the usual telegraphic form[9]. An intuitive understanding of the definition of a '*basin of attraction*' is probably sufficient for a complete comprehension of the result that is of interest here - provided there is reasonable familiarity with the definition and properties of *Turing Machines* (or partial recursive functions or equivalent formalisms encapsulated by the *Church-Turing Thesis*).

Definition 1 *The Initial Value Problem (**IVP**) for an Ordinary Differential Equation (**ODE**) and **Flows**.*

Consider a differential equation:

$$x = f(x) \tag{1}$$

where x is an unknown function of $t \subset I$ *(say, t: time and I an open interval of THE REAL LINE) and f is a given function of x. Then, a function x is a solution of (7) on the OPEN INTERVAL I if:*

$$x(t) = f(x(t)), \forall t \subset I \tag{2}$$

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[9] In the definition of a dynamical system given below we are not striving to present the most general version. The basic aim is to lead to an intuitive understanding of the definition of a basin of attraction so that the main theorem is made reasonably transparent. Moreover, the definiton given below is for scalar ODEs, easily generalizable to the vector case.

The initial value problem (ivp) for (1) is, then, stated as:

$$\dot{x} = f(x), \qquad x(t_0) = x_0 \tag{3}$$

and a solution x(t) for (3) is referred to as a solution through x_0 *at* t_0*. Denote x(t) and* x_0*, respectively, as:*

$$\varphi(t, x_0) = x(t), \text{ and } \varphi(0, x_0) = x_0 \tag{4}$$

where $\varphi(t, x_0)$ *is called the* ***flow*** *of* $\dot{x} = f(x)$*.*

Definition 2 ***Dynamical System***

if f is a C^1 *function (i.e., the set of all differentiable functions with continuous first derivatives), then the* ***flow*** $\varphi(t, x_0)$, $\forall t$*, induces a* ***map*** *of* ***U*** $\subset \mathbb{R}$ *into itself, called a* C^1 ***dynamical system on*** $\mathbb{R}$:

$$x_0 \mapsto \varphi(t, x_0) \tag{5}$$

if it satisfies the following (one-parameter group) properties:

- $\varphi(0, x_0) = x_0$
- $\varphi(t + s, x_0) = \varphi(t, \varphi(s, x_0))$, $\forall t$ *& s. whenever both the l.h and r.h side maps are defined;*
- $\forall t$, $\varphi(t, x_0)$ *is a* C^1 *map with a* C^1 *inverse given by:* $\varphi(-t, x_0)$*;*

Remark 3 *A geometric way to think of the connection between a* ***flow*** *and the induced* ***dynamical system*** *is to say that the flow of an* ***ODE*** *gives rise to a dynamical system on* $\mathbb{R}$*.*

Remark 4 *It is important to remember that the* ***map*** *of* ***U*** $\subset \mathbb{R}$ *into itself may* ***not*** *be defined on all of* $\mathbb{R}$*. In this context, it might be useful to recall the distinction between partial recursive functions and total functions in classical recursion theory.*

Definition 5 ***Invariant set***

A set (usually compact) $S \subset U$ *is* ***invariant*** *under the flow* $\varphi(.,.)$ *whenever* $\forall t \subset \mathbb{R}$,

$\varphi(.,.) \subset S$.

Definition 6 ***Attracting set***

A closed invariant set $A \subset U$ *is referred to as the* ***attracting set*** *of the* ***flow*** $\varphi(t, x)$ *if* $\exists$ *some neighbourhood V of A, s.t* $\forall x \subset V$ & $\forall t \geq 0, \varphi(t, x) \subset V$ *and:*

$$\varphi(t, x) \to A \text{ as } t \to \infty \tag{6}$$

Remark 7 *Recall that in dynamical systems theory contexts the attracting sets are considered the* ***observable*** *states of the dynamical system and its flow.*

Definition 8 *The basin of attraction of the attracting set A of a flow, denoted, say, by* Θ_A*, is defined to be the following set:*

$$\Theta_A = \bigcup_{t \le 0} \varphi_t(V) \tag{7}$$

where: $\varphi_t(.)$ *denotes the flow* $\varphi(.,.), \forall t$.

Remark 9 *Intuitively, the basin of attraction of a flow is the* ***set of initial conditions*** *that eventually leads to its attracting set - i.e., to its limit set (limit points, limit cycles, strange attractors, etc). Anyone familiar with the definition of a Turing Machine and the famous Halting problem for such machines – or, alternatively, Rice's theorem – would immediately recognise the connection with the definition of basin of attraction and suspect that our main result will be obvious.*

Definition 10 ***Dynamical Systems capable of Computation Universality:***

A dynamical system capable of computation universality is one whose defining initial conditions can be used to program and simulate the actions of any arbitrary Turing Machine, in particular that of a Universal Turing Machine.

Proposition 11 *Dynamical systems characterizable in terms of limit points, limit cycles or `chaotic' attractors, called `elementary attractors', are* ***not*** *capable of universal computation.*

Proof. Essentially because the basin of attraction of such dynamical systems are recursive. ■

Proposition 12 *Strictly linear dynamical systems are incapable of computation universality.*

Proof. Strictly linear dynamical systems are formally equivalent to finite automata. ■

Proposition 13 *Only dynamical systems whose basins of attraction are poised on the boundaries of elementary attractors are capable of universal computation.*

Proof. Essentially by construction ■

Theorem 14 *There is no effective procedure to decide whether a given observable trajectory is in the basin of attraction of a dynamical system capable of computation universality*

Proof. The first step in the proof is to show that the basin of attraction of *a dynamical system capable of universal computation is recursively enumerable but not recursive. The second step, then, is to apply Rice's theorem* to the problem of membership decidability in such a set.

First of all, note that the basin of attraction of a dynamical system capable of universal computation is recursively enumerable. This is so since trajectories belonging to such a dynamical system can be effectively listed simply by trying out, systematically, sets of appropriate initial conditions.

On the other hand, such a basin of attraction is not recursive. For, suppose a basin of attraction of a dynamical system capable of universal computation is recursive. Then, given arbitrary initial conditions, the Turing Machine corresponding to the dynamical system capable of universal computation would be able to answer whether (or not) it will halt at the particular configuration characterising the relevant observed trajectory.

This contradicts the unsolvability of the Halting problem for Turing Machines.

Therefore, by Rice's theorem, *there is no effective procedure to decide whether any given arbitrary observed trajectory is in the basin of attraction of such recursively enumerable but not recursive basin of attraction*. Only dynamical systems whose basins of attraction are poised on the boundaries of elementary attractors are capable of universal computation. ■

Remark 15 *There is a classic mathematical `fudge' in our proof of the recursive enumerability of the basin of attraction: how can one try out, `systematically', the set of uncountable initial conditions lying in the appropriate subset of* $\boldsymbol{R}$*? Of course, this cannot be done and* the theorem is given just to give an idea of the problem that we want to consider.

Claim 16 ***Only dynamical systems capable of computation universality are dynamically complex in the sense of Day-Rosser.***

The next natural question would be whether such systems exist. The reason – mentioned above, in section two, is that the Day-Rosser definition is a `negative' one: a dynamically complex system is defined in terms of *what it does not do*. This is like checking for *non-membership* of a *recursively enumerable* but *non-recursive* set, which is impossible. Instead, the best way to answer such a question would be to *demonstrate by construction* the existence of such a system.

Keeping the framework and the question in mind, one way to proceed would be to constructivise the basic IVP problem for ODEs and then the theorem can be applied consistently. Alternatively, it is possible to work with ODE's within the framework of computable analysis and use *recursive (in)separability* judiciously to generate non-recursive solutions that are intrinsically complex to compute without *oracles*. Both of these methods will require the development of a wholly `unfamiliar' set of concepts within an even more strange mathematical framework. It will require too much space and time to do so within the scope of this paper. Instead, we shall adopt a slightly devious method.

Consider the following Generalized Shift (GS) map ([27], [28]):

$$\Phi : \wp \to \sigma^{F(\wp)}[\wp \oplus G(\wp)] \tag{8}$$

Where:

$\wp$:(bi-infinite) symbol sequence;

F : mapping from a finite subset of $\wp$ to the integers;

G : mapping from a finite subset of $\wp$ into $\wp$;

σ : a shift operator;

The given `finite subset of $\wp$ ', on which F and G operate is called the *domain of dependence* (DOD).

Let the given symbol sequence be, for example:

$$\wp = \{\ldots p_{-1} p_0 p_1 \ldots\} \tag{9}$$

Then:

$\wp \oplus G(\wp) \Rightarrow$ replace DOD by G($\wp$).

$\sigma^{F(\wp)} \Rightarrow$ *shift* the sequence left or right by the amount F($\wp$).

Remark 17 *In practice, a GS is implemented by denoting a distinct position on the initially given symbol sequence as, say, p_0 and placing a `reading head' over it. It must also be noted $p_i = \wp, \forall i = 1, 2, \ldots$ could, for example, denote whole words from an alphabet, etc., although in practice it will be 0,1 and $\bullet$ (`dot'). The 'dot' will be signify that the `reading head' will be placed on the symbol to the right of it.*

The following results about Generalized Shift maps are relevant for this discussion:

Proposition 18 *Any GS is a **nonlinear** (in fact, **piecewise linear**) dynamical system capable of universal computation; hence they are universal dynamical systems and are equivalent to some constructible Universal Turing Machine.*

Thus the GS is capable of universal computation and it is **minimal** in a precisely definable sense (see [27] and [28] for full details). It is also possible to construct, for each such generalized shift dynamical system[10], an equivalent UTM that can simulate its dynamics, for sets of initial conditions.

Next, consider the definition of the Kolmogorov complexity of a finite object (Kolmogorov, 1968 [[21]], p. 465):

$$K_\phi(y \mid x) = \begin{cases} \min_{\phi(p,x)=y} l(p) \\ \infty \rightarrow \nexists p \ s.t. \ \phi(p,x) = y \end{cases} \tag{10}$$

Where: $\phi(p,x) = y$: a partial recursive function or, equivalently (by the Church-Turing Thesis), a Turing Machine – the `method of programming' – associating a (finite) object *y* with a *program p* and a (finite) object *x*; the minimum is taken over all programs capable of generating *y*, on input *x*, to the partial recursive function, *p*. Consider the above (minimal) universal dynamical system as canonical for any question about membership in attracting sets, *A*. What is the complexity of $K_U(p \mid x)$? By definition it should be:

$$K_U(y \mid x) = \begin{cases} \min_{U(p,x)=y} l(p) \\ \infty \rightarrow \nexists p \ s.t. \ \phi(p,x) = y \end{cases}$$

The meaning, of course, is: the minimum over all programs, *p*, implemented on *U*, with the given initial condition, *x*, which will stop at the halting configuration, *y*.

Unfortunately, however, $K_U(y \mid x)$ is *a non-recursive real number!*

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[10] They can also encapsulate smooth dynamical systems in a precise sense. We have described the procedure, summarising a variant of Chris Moore's approach, in [41], Chapter 4.

How can we decide, algorithmically, whether any observed trajectory is generated by the dynamics of a system capable of computation universality? Consider the observable set of the dynamical system, $y \subset \mathcal{A}$; given the UTM, say U, corresponding to $\wp$; the question is: for what set of initial conditions, say *x*, is *y* the halting state of U. Naturally, by the theorem of the *unsolvability of the Halting problem*, this is an *undecidable* question.

Remark 19 *Why is it important to show the existence of the minimal program? Because, if the observed y corresponds to the minimal program of the dynamical system, i.e., of* U*, then it is capable of computation universality; if there is no minimal program, the dynamical system is not interesting! A monotone decreasing set of programs that can be shown to converge to the minimal program is analogous to a series of increasingly complex finite automata converging to a TM. What we have to show is that there are programs converging to the minimal program from above and below, to the border between two basins of attractions.*

Shortly after Kolmogorov's above paper was published, Zvonkin and Levin, [44], p.92, Theorem 1.5,b, provided the result and proof that rationalises the basic principle of the *computable approximation* to the uncomputable $K_{\varphi}(y\,|\,x)$. The significant relevant result is:

Theorem 20 *Zvonkin-Levin*

∃ *a general recursive function* $H(t,x)$, *monotonically decreasing in t, s.t:*

$$\lim_{t \to \infty} H(t,x) = K_{\varphi}(y\,|\,x) \tag{11}$$

Remark 21 *This result guarantees, the existence of `arbitrarily good upper estimates' for* $K_{\varphi}(y\,|\,x)$*, even although* $K_{\varphi}(y\,|\,x)$ *is uncomputable. We are not sure this is a claim that is constructively substantiable*[11]*. How can a noncomputable function be approximated? If any one noncomputable function can be approximated uniformly, then by `reduction' it should be possible, for example, to `approximate', say, the Busy Beaver function. We suspect an intelligent and operational interpretation of the Zvonkin-Levin theorem requires a broadening of the notion of `approximation'.*

But formally, at least, we can obviate the above result on the algorithmic impossibility of inferring, from observable trajectories, whether they have been generated by a dynamical system capable of computation universality. In fact, the formality is encapsulated elegantly and effectively in the way Jorma Rissanen

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[11] Our view on this is further strengthened by some of the remarks in [9], particularly, p.163, where one reads (italics added):

> "The shortest program is not computable, although as more and more programs are shown to produce the string, the estimates from above of the Kolmogorov complexity *converge to the true Kolmogorov complexity*, (the problem, of course, is that *one may have found the shortest program and never know that no shorter program exists*).

These remarks border on the metaphysical! How can one *algorithmically* approximate to a true value that which cannot be known *algorithmically* – by definition?

developed the theory of stochastic complexity in Rissanen (1978, 1986). Thus, we can try to approximate to the undecidable by a monotone computable process; i.e., we can approximate to the dynamical system capable of computation universality by a sequence of observation on simpler dynamical systems. Unfortunately, however, the melancholy fact noted in the last footnote may haunt the empiricist forever!

What we have demonstrated can be summarized as a *Generalised Day-Rosser Proposition* as follows.

Proposition 22 *Generalised Day-Rosser Proposition*

Dynamical systems that are dynamically complex in the sense of the Day-Rosser definition are computation universal. There exist *constructible* dynamical systems that are definably minimal and capable of computation universality. Their Kolmogorov complexity can be defined unambiguously, but cannot be computed exactly. The inference problem for such systems possesses strong undecidability properties. However, the Kolmogorov complexity can be approximated by a sequence of monotone decreasing general recursive functions.

4. Beyond Dynamic Complexities - Towards an Algorithmic Formalization of Emergence

"Professor Leontief ... maintains that we may utilize dynamical systems that are *unstable throughout* and cites *capitalism* as an example."

Goodwin, 1953, p. 68; italics added ([17])

Day-Rosser dynamical systems are `*unstable throughout*' in the sense of being situated on the borderline between *decidable* dynamical systems. Their Kolmogorov complexities can be defined and, although uncomputable, can be approximated *effectively*.

Moreover, the Day-Rosser definition is independent of dimension. Much of the hype surrounding complex dynamics is against the backdrop of the necessity of systems with interacting fundamental units, coupled nonlinearly. All of the formalism in the main section of this paper has been in terms of a scalar dynamical system. A one-dimensional, piece-wise linear, dynamical system was shown to satisfy the Day-Rosser characterization of dynamic complexity.

How, from the dynamic complexity of a one-dimensional dynamical system could we generate emergent behaviour, however conceived?

We have a simple – even simplistic – answer to this seemingly deep or difficult question, at least from the point of view of the formalism developed above. We can define emergent behaviour as that exhibited by the behaviour of a dynamical system in its ***transition*** from one that is incapable of computation universality to one that is capable of such universal behaviour. In other words emergent behaviour is that which is manifested in *the transition from the computable to the incomputable* -- or, from the decidable to the undecidable. This is the kind of approach that has been suggested by Barry Cooper in his recent work (see Cooper, 2006). We endorse and accept this vision, and our formalism was inspired by having this vision as the backdrop for our interpretation of the classics of the emergence literature, in particular the pioneering works of John Stuart Mill ([26]), George Henry Lewes ([22]) and C. Lloyd Morgan ([23]).

The concept of *emergence*, crucial in the modern sciences of complexity, came to have its current connotations as a result of these (and a few other) clearly identifiable sequence of classic works by this trio (and one or two others, as indicated below). A representative sample of key observations, pertaining to this notion, is given below, partly because the economic literature on *Complexity*, of whatever hue, has paid no attention whatsoever to these origins. In particular, not even a meticulous and fastedious scholar as Hayek, in [18], often referred to as a fountainhead for the role of emergence in the cognitive sciences, has made any reference to Lloyd Morgan and thereby, mysteriously, omitted Mill, of whom he was one of the great admirers and students. A representative sample of crucial definitions, in these classics, may go part of the way towards substantiating the vision we are advocating, *especially the one by Lewes*, the man who introduced the word *'emergent'*, from which Lloyd Morgan derived *'emergence'*:

> "[T]here are laws which, like those of chemistry and physiology, owe their existence to .. *heteropathic laws*… . The Laws of Life will never be deducible from the mere laws of the ingredients, but the prodigiously *complex* Facts of Life may all be deducible from comparatively *simple* laws of life;…"
>
> John Stuart Mill (1890) Bk.III, Ch.VI, p.269; italics added.
>
> "Thus, although each effect is the resultant of its components, the product of its factors, we cannot always trace *the steps of the process*, so as to see in the product the mode of operation of each factor. In this latter case, I propose to call the effect *an emergent*. It arises out of the combined agencies, but in a form which does not display the agents in action."
>
> George Henry Lewes (1891) Problem V, Ch.III, p.368, italics added.
>
> "The concept of *emergence* was dealt with .. by J.S.Mill in his Logic .. under the discussion of *`heteropathic laws'* in causation. The word *`emergent'* as contrasted with *`resultant,'* was suggested by G.H.Lewes in his Problems of Life and Mind'. … . What makes emergents emerge? .. *What need [is there] for a directive Source of emergence. Why should it not proceed without one?"*
>
> C. Lloyd Morgan (1927), pp. 2, 32; italics added.

The trio of Mill, Lewes and Lloyd Morgan, together with C. D. Broad (1929) and Samuel Alexander (1920) made up what has come to be called the *'British Emergentist'* school. Their rise and fall has been eloquently and almost persuasively argued by Brian McLaughlin (1992), standing on the shoulders of the philosophical critiques of the 1920s, launched primarily by Stephen Pepper (1926), W. T. Stace (1939) and Charles Bayliss (1929). But none of these critiques of the pioneers were blessed with the visions or, in the case of McLaughlin, knowledge of, algorithmic mathematics. Our vision is informed and underpinned by an algorithmic epistemology, for dynamics, inference and formalization. From such a standpoint, the *British Emergentists* were prescient in their approach to the formalization of emergence, coupled to the dialectic between the simple and the complex, in a natural dynamic context. They *rise and rise*; there was **never any fall** of the *British Emergentisits*!

Emergence, order, self-organisation, turbulence, induction, evolution, (self-organized) criticality, adaptive, non-linear, networks, irreversible, non-equilibrium are some of the 'buzz' words, in various permutations and combinations, that characterize the conceptual underpinnings of the 'new' *sciences of complexity* that seem to pervade some of the frontiers in the natural, social and even the human sciences. Not since the heyday of *Cybernetics* and the more recent ebullience of chaos applied to a theory of everything, has a concept become so prevalent and pervasive in almost all fields, from Physics to Economics, from Biology to Sociology, from Computer Science to Philosophy as *Complexity* seems to have become. Even though lip service is paid to the poverty of axiomatics and the deductive method, almost without exception the mathematics employed by the practitioners of the new sciences of complexity is the classical variety: real analysis, founded on axiomatic set theory, supplemented by one or another variety of the axiom of choice. These theoretical technologies are, *ab initio*, non-algorithmic. This dichotomy – or is it, perhaps, a schizophrenia – pervades the work of, for example, those advocates of agent-based modelling who claim that the generation of patterns mimicking data at one level, based on foundations at a simpler level, is the best approach to the study of emergence, complexity and dynamics.

We disagree.

Einstein, perceptively, summarized a particular epistemology that he followed in his scientific endeavours:

> "If, then, it is true that the axiomatic basis of theoretical physics cannot be extracted from experience but must be freely invented, can we ever hope to find the right way? .... I answer without hesitation that there is, in my opinion, a right way, and that we are capable of finding it. Our experience hitherto justifies us in believing that nature is the realization of the simplest conceivable mathematical ideas. I am convinced that we can discover by means of purely mathematical constructions the concepts and the laws connecting them with each other, which furnish the key to the understanding of natural phenomena."
>
> Einstein in his **Herbert Spencer Lecture**, '*On the Methods of Theoretical Physics'*, given at Oxford University on June 10, 1933; d in:([29]), pp.136-7; italics added.

Economists, too, advocating a complexity vision would have the economic theorist discard the axiomatic, deductive, method altogether. These economists also claim and assert that the epistemological and methodological poverty of so-called `deductive mathematics' in an epoch where the ubiquity of the digital computer makes what they claim to be 'inductive and abductive methods' far superior in tackling formally intractable economic problems. For a supreme theorist like Einstein, grappling with no less empirically unruly phenomena than those faced by the economist, an epistemological underpinning, like the one explicitly acknowledged above, provides a justification for the methods of axiomatic, mathematical, theorizing. Any advocacy of a new vision and its methods, without an epistemological basis to underpin them, runs the risk of having to be hoisted by its own petard and, therefore, of living dangerously. In a certain clearly substantiable sense, Richard Day and Barkley Rosser stand on the giant shoulders of Charles Sanders Peirce, one in that line of great American mathematical epistemologists, descending towards Warren McCulloch and Norbert Wiener, in the immediate epoch before our own times.

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